Solving The Rocket Equation

Pascal Bakker and Colton Burnham

Wentworth Institute of Technology

bakkerp@wit.edu
burnhamc@wit.edu

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Overview

Introduction to the Equation

- Launching a rocket in to space is not a matter of simple projectile motion
- Involves a further understanding of physics and calculus
- As a rocket is launched it burns fuel
- This reduces its weight and effects the flight

"The Rocket Equation"

These factors are taken in to account and the final equation is modeled by:

$$\frac{dv}{dt} = \frac{Ru_{ex}}{m_i - (Rt)} - g \tag{1}$$

where:

- $(R * u_{ex}) = F_{th}$ — Thrust
- R —— Burn rate of the fuel
- u_{ex} — Speed of the exhausted fuel relative to the rocket
- m_i —- Initial mass of the rocket containing fuel
- t —— Time
- g —— Acceleration due to gravity

Exact Solutions

Many ways to approximate the solution to a differential equation

- Euler's
- Runge Kutta (RK2)

Must be compared to an exact solution to verify the validity. The exact solution to e.q (1) is

$$v(t) = -u_{ex} ln(1 - \frac{Rt}{m_i}) - (gt) \tag{2}$$

and can be solved with simple algebra once the constants and time have been inputed.

Exact Solution Cont.

Since velocity is understood as the change in position over a change in time, or in other words the integral of velocity, we can view e.q (2) as

$$\frac{dx}{dt} = -u_{ex} ln(1 - \frac{Rt}{m_i}) - (gt) \tag{3}$$

and after basic calculus the exact solution to the rocket's position becomes

$$x(t) = u_{ex} \frac{m_i(1 - \frac{Rt}{m_i})(1 - ln(1 - \frac{Rt}{m_i}))}{R} - g\frac{t^2}{2} \tag{4}$$

Proofs

- The Euler and Runge Kutta methods can only approximate the solution
- We need to get the proofs to verify these approximations

Velocity Proof

$$\frac{dv}{dt} = \frac{Ru_{ex}}{m_i - Rt} - g \quad (5)$$

$$Ru_{ex} \int \frac{1}{m_i - Rt} - gdt \quad (6)$$

Substitute: $u = m_i - Rt$ $du =$ -Rdt $dt = \frac{du}{-R}$

$$-u_{ex} \int \frac{1}{u} du - gt \quad (7)$$

$$-u_{ex} ln(u) - gt \quad (8)$$

$$v(t) = -u_{ex} ln(m_i - Rt) - gt \quad (9)$$

Unit Analysis for Velocity

$$\frac{dv}{dt} = \frac{Ru_{ex}}{m_i - Rt} - g \tag{10}$$

$$\frac{\frac{m}{s}}{s} = \frac{(\frac{kg}{s})(\frac{m}{s})}{kg - \frac{kg}{s}(s)} - \frac{m}{s^2} \tag{11}$$

$$\frac{m}{s^2} = (\frac{(kg)(m)}{s^2})(\frac{1}{kg}) - \frac{m}{s^2} \tag{12}$$

$$\frac{m}{s^2} = \frac{m}{s^2} - \frac{m}{s^2} \tag{13}$$

$$\frac{m}{s^2} = \frac{m}{s^2} \tag{14}$$

Position Proof

To check the accuracy of the Euler Method, we must get the exact position for the rocket.

$$v(t) = -u_{ex} ln(1 - \frac{Rt}{m_i}) - gt \tag{15}$$

$$x(t) = \int -u_{ex} ln(1 - \frac{Rt}{m_i}) - gtdt \tag{16}$$

Substitute: $u = 1 - \frac{Rt}{m_i}, du = \frac{-R}{m_i}, dt = \frac{-m_i}{R} du$

$$x(t) = -u_{ex} \int ln(u) \frac{-m_i}{R} - \frac{gt^2}{2} \tag{17}$$

Position Proof Continued

$$x(t) = -u_{ex} \int ln(u) \frac{-m_i}{R} - \frac{gt^2}{2} \tag{18}$$

$$x(t) = (\frac{m_i u_{ex}}{R})(u ln(u) - \frac{Rt}{m_i}) - \frac{gt^2}{2} \tag{19}$$

$$x(t) = (\frac{m_i u_{ex}}{R})((1 - \frac{Rt}{m_i}) ln(1 - \frac{Rt}{m_i}) - \frac{Rt}{m_i}) - \frac{gt^2}{2} \tag{20}$$

Unit Analysis for Position

$$x(t) = u_{ex}\frac{m_i(1-\frac{Rt}{m_i})(1-ln(1-\frac{Rt}{m_i}))}{R} - g\frac{t^2}{2} \tag{21}$$

$$m = \frac{m}{s}(\frac{kg(\frac{\frac{kg}{s}(s)}{kg})ln(\frac{\frac{kg}{s}(s)}{kg})}{\frac{kg}{s}}) - \frac{m}{s^2}(s^2) \tag{22}$$

$$m = \frac{m}{s}(\frac{kg}{\frac{kg}{s}}) - m \tag{23}$$

$$m = \frac{m}{s}(s) - m \tag{24}$$

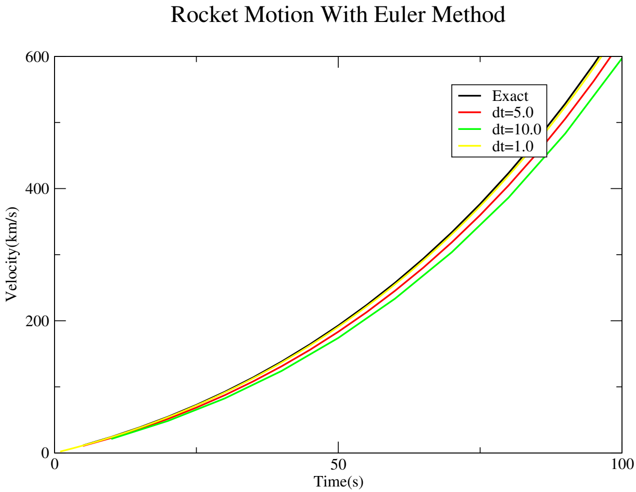
$$m = m - m \tag{25}$$

$$m = m \tag{26}$$

Euler Method Explained

- Get Initial Conditions and Time Step
- The independent variable at point a is equal to the previous value at the previous time step plus the the derivative value at the previous time step
- Smaller time step = more accurate
- Only requires 1 for loop in the program

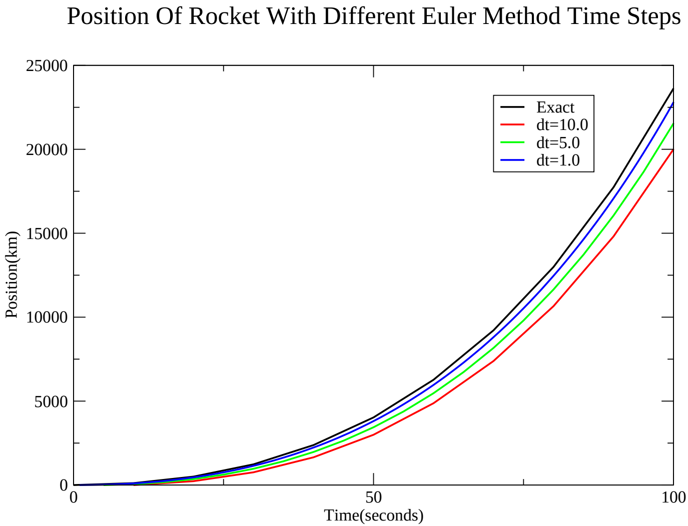
Velocity - Euler's Method



Velocity explained

- Since the velocity follows the function $-ln(1-x)$ the rocket will continue to accelerate
- This function ignores the amount of fuel in the rocket ship.
- Is ideal model when burn rate is constant

Position - Euler's Method



- As the atmospheric pressure decreases, the pressure on the rocket decreases
- The longer the ship is in motion, the more efficient the rocket is.

Runge Kutta Method Explained

$$\frac{df}{dt} = f(y_n, t_n) \quad (27)$$

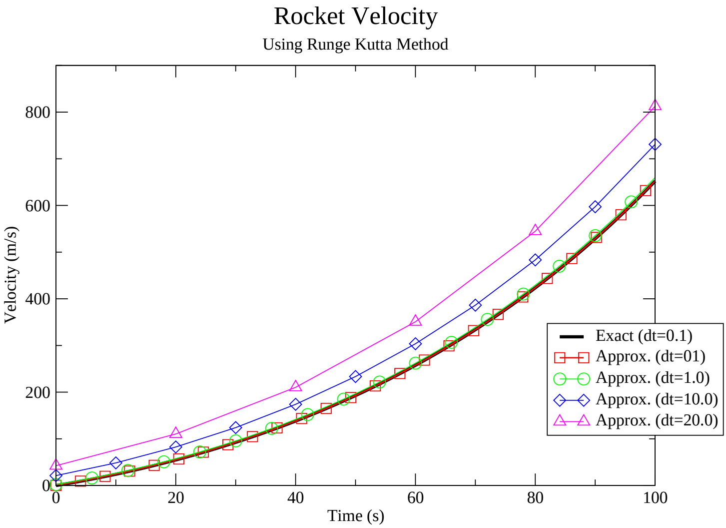
$$k1 = \Delta t f(y_n, t_n) \quad (28)$$

$$k2 = \Delta t(y_n + \frac{1}{2}k1, t_n + \frac{1}{2}\Delta t) \quad (29)$$

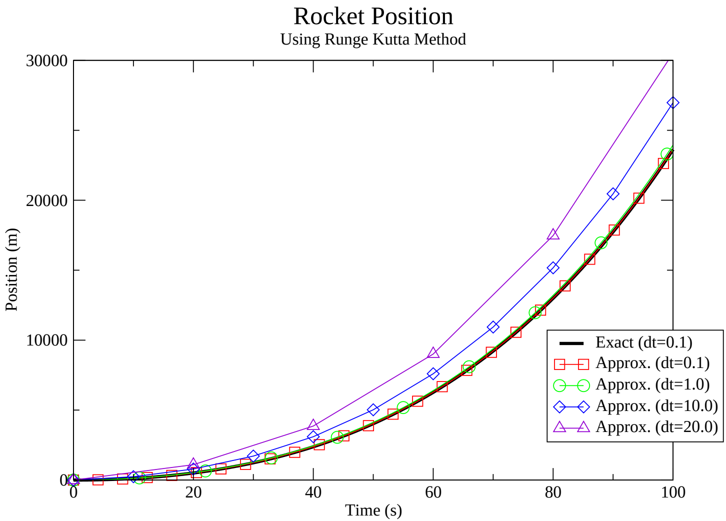
$$y_{n+1} = y_n + k2 \quad (30)$$

- n represents the current term, while $n+1$ represents the next
- For our calculations, $\frac{df}{dt}$ is $\frac{dv}{dt}$ and $\frac{dx}{dt}$

Velocity - Runge Kutta Method



Position - Runge Kutta Method



Additional Calculations

$$u_{ex} = \frac{F_{th}}{R} \tag{31}$$

$$m_f = payload(m_i) \tag{32}$$

$$t_b = \frac{m_i - m_f}{R} = \frac{m_i - (payload(m_i))}{R} \tag{33}$$

For our initial conditions the results are:

- Speed of exhausted fuel $= u_{ex} = 2456.6474\ \frac{m}{s}$
- Final Mass $= m_f = 769500.0306\ kg$
- Burn time $= t_b = 150.3251\ seconds$

Computational Aspect

- Calculating Differential Equations via Euler and RK method
- Can be easily modified for different initial variables.
- Solutions calculated continuously over time provided a user input time step
- Calculations by hand would require solving each individual solution at each time variable
- Computationally, this is done using loops that solve the equations at each iteration of time

References

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Questions?